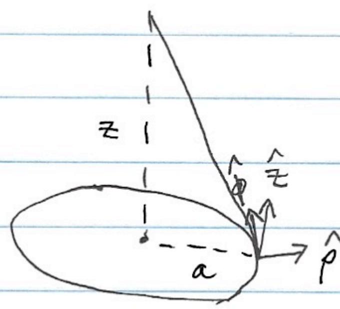
Jackson 5.7(a)

$$d\vec{B} = kI \frac{(d\vec{l} \times \vec{x})}{|\vec{x}|^3}$$

More rigorously, $$d\vec{B} = kI \frac{d\vec{l} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}.$$

$$d\vec{l} = a\,d\phi\,\hat{\phi} \qquad \vec{x} - \vec{x}' = z\hat{z} - a\hat{\rho}$$



$$d\vec{l} \times (\vec{x} - \vec{x}') = a\,d\phi\,[\hat{\phi} \times (z\hat{z} - a\hat{\rho})]$$
$$= a\,d\phi\,[z\hat{\rho} + a\hat{z}]$$

The $\hat{\rho}$ component will cancel each other around the circle.

$$\Rightarrow \vec{B} = kI \int_0^{2\pi} \frac{a\,d\phi\,[a\hat{z}]}{(a^2+z^2)^{3/2}}$$

$$= 2\pi k I \frac{a^2}{(a^2+z^2)^{3/2}} \hat{z}$$

$$= \boxed{\frac{\mu_0 I}{2} \frac{a^2}{(a^2+z^2)^{3/2}} \hat{z}}$$

Jackson 5.7 (c)

We have from part (b)

$$B_z(z) = \left(\frac{\mu_0 I a^2}{d^3}\right)\left[1 + \frac{3(b^2-a^2)z^2}{2d^4} + \frac{15(b^4-6b^2a^2+2a^4)}{16d^8}z^4 + \cdots\right]$$

From exercise 5.4:

$$B_z(\rho, z) \simeq B_z(0, z) - \left(\frac{\rho^2}{4}\right)\left[\frac{d^2 B_z(0,z)}{dz^2}\right] + \cdots$$

Up to second order we already have from

$$B_z(\rho=0, z) \simeq \frac{\mu_0 I a^2}{d^3} + \left(\frac{\mu_0 I a^2}{d^3}\right)\left(\frac{3(b^2-a^2)}{2d^4}\right)z^2$$

from

It's a simple matter to expand $B_z(\rho, z=0)$ to obtain

$$B_z(\rho, z=0) \simeq \frac{\mu_0 I a^2}{d^3} - \left(\frac{\rho^2}{4}\right)\left(\frac{\mu_0 I a^2}{d^3}\right)\frac{3(b^2-a^2)}{d^4}$$

Denoting $\sigma_0 = \frac{\mu_0 I a^2}{d^3}$, $\sigma_2 = \left(\frac{\mu_0 I a^2}{d^3}\right)\left(\frac{3(b^2-a^2)}{2d^4}\right)$, we have

$$\boxed{B_z(\rho, z) \simeq \sigma_0 + \sigma_2\left(z^2 - \frac{\rho^2}{2}\right)}$$

Carrying the same procedure with the expansion

$$B_\rho(\rho, z) \approx -\left(\frac{\rho}{2}\right)\left[\frac{\partial B_z(0,z)}{\partial z}\right] + \cdots$$

from exercise 5.4, we obtain.

$$\boxed{B_\rho \approx -\sigma_2 z \rho}$$

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2.2.2024